On a sufficient condition for p-valently starlikeness

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Let A(p) be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in N = \{1,2,3,\ldots\})$$

which are regular in $D = \{ z \mid |z| < 1 \}$.

A function f(z) in A(p) is said to be p-valently starlike iff

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0 \qquad \text{in } D.$$

We denote by S(p) the subclass of A(p) consisting of functions which are p-valently starlike in D.

THEOREM. Let $f(z) \in A(p)$ and assume that

(1) $$\left| \arg \frac{f'(z)}{z^{p-1}} \right| < \frac{\pi}{2}\alpha \qquad \text{in } D$$

and

(2) $$\left(\mathrm{Im}\, \frac{f'(z)}{z^{p-1}} \right)\left(\mathrm{Im}\, e^{-i\beta} z \right) \neq 0$$

for $z \in D(\beta) = \{ z \mid |z| < 1,\ z \neq 0 \text{ and } (\arg z - \beta)(\arg z - \beta - \pi) \neq 0 \}$ where α and β are real numbers, $0 < \alpha \leqq 1$ and $0 \leqq \beta < \pi$.

Then we have

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}\alpha \qquad \text{in } D$$

and therefore f(z) is p-valently starlike in D or $f(z) \in S(p)$.

PROOF. Applying the same metod as in the proof of Ruscheweyh [1. p.142], we have

$$\frac{f(z)}{zf'(z)} = \int_0^1 \frac{f'(tz)}{f'(z)}\, dt$$

$$= \frac{z^{p-1}}{f'(z)} \int_0^1 t^{p-1} \frac{f'(tz)}{(tz)^{p-1}}\, dt, \qquad z \in D$$

and it follows that

$$\arg t^{p-1} \frac{f'(tz)}{(tz)^{p-1}} = \arg \frac{f'(tz)}{(tz)^{p-1}}. \tag{3}$$

From the assumption (1) and (2), and from (3), if we have

$$0 < \arg \frac{f'(z)}{z^{p-1}} < \frac{\pi}{2}\alpha ,$$

then the integral

$$\int_0^1 t^{p-1} \frac{f'(tz)}{(tz)^{p-1}}\, dt$$

lies in the same convex sector $\left\{ z \mid 0 < \arg z < \frac{\pi}{2}\alpha \right\}$ and by the same reason as the above, if we have

$$0 > \arg \frac{f'(z)}{z^{p-1}} > -\frac{\pi}{2}\alpha$$

then we have

$$0 > \arg \left(\int_0^1 t^{p-1} \frac{f'(tz)}{(tz)^{p-1}}\, dt \right) > -\frac{\pi}{2}\alpha .$$

Therefore we have

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}\alpha \qquad \text{in } D.$$

This shows that $f(z)$ is p-valently starlike in D.

From the THEOREM , we easily have the following corollaries:

COROLLARY 1. Let $f(z) \in A(p)$ and assume that $0 < \alpha \leqq 1$ and

$$\left| \arg \frac{f'(z)}{z^{p-1}} \right| < \frac{\pi}{2}\alpha \qquad \text{in } D$$

and $f'(z)/z^{p-1}$ is typically real in D.

Then $f(z)$ belongs to $S(p)$ and

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}\alpha \qquad \text{in } D.$$

COROLLARY 2. Let $f(z) \in A(p)$ and assume that $\mathrm{Re}(f'(z)/z^{p-1}) > 0$ in D and $f'(z)/z^{p-1}$ is typically real in D. Then $f(z)$ belongs to $S(p)$ or

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0 \qquad \text{in D.}$$

COROLLARY 3. Let $f(z) \in A(1)$ and assume that $f'(z)$ is typically real in D and satisfies

$$|\arg f'(z)| < \frac{\pi}{2}\alpha \qquad \text{in D.}$$

Then $f(z)$ is univalently starlike and

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\alpha \qquad \text{in D}$$

Reference

[1] S. Ruscheweyh, Coefficient conditons for starlike functions, Glasgow Math. J., 29(1987), 141-142.